
Kernel-Induced Statistical Representations for Continuous Stiffness Field Inversion via Karhunen–Loève Expansion

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Abstract

Learning representations of continuous physical fields from sparse observations remains challenging due to the mismatch between high-dimensional spatial variations and limited measurement information. In this work, continuous stiffness inversion is formulated as a structured latent representation learning problem rather than a direct field regression. A Karhunen–Loève (KL) expansion derived from a spatial covariance kernel is introduced as a mathematical representation of continuously varying stiffness fields, projecting the spatial field into a compact coefficient space. The resulting latent variables provide a constrained parameterization of admissible field variations and reduce the ambiguity of the inverse mapping from dynamic responses to spatial properties. By learning a direct mapping from sparse frequency response functions (FRFs) to KL coefficients, the proposed framework reconstructs continuous stiffness fields without intermediate modal feature extraction. Experiments on a simply supported Euler–Bernoulli beam benchmark demonstrate accurate field reconstruction (Field Error = 4.06%, $R^2 = 0.9779$) under 5% measurement noise. The results illustrate how statistical field representations can facilitate continuous inference of physical parameters from sparse dynamic observations.

1 Introduction

Structural stiffness is a fundamental physical parameter governing the dynamic behavior, load-bearing capacity, and structural integrity of engineering systems [1–4]. Reconstructing continuous stiffness fields from dynamic responses enables quantitative evaluation of structural degradation, localization of weak regions, and integration with forward structural analysis for engineering assessment [5, 6]. Consequently, continuous stiffness reconstruction has become an important problem in structural health monitoring (SHM), digital twins, and predictive maintenance, where spatially varying structural properties must be identified from incomplete observations [7–10].

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Traditional stiffness inversion relies on modal analysis [11, 12] to extract discrete parameters from sparse sensor measurements [13–15]. However, low-order modal truncation and high noise sensitivity limit their spatial resolution, rendering continuous field reconstruction inherently ill-posed. Although recent machine learning methods attempt to bridge this gap through Bayesian frameworks [16, 17], model updating [18], or deep learning [19–21], they still suffer from two primary limitations:

First, **explicit feature extraction causes lossy observation representations**. Relying on intermediate modal or spectral features creates a predefined information bottleneck, discarding localized response variations while propagating extraction uncertainties in downstream inference [22, 23].

Second, **continuous field reconstruction remains severely ill-posed**. The mapping from sparse dynamic observations to distributed stiffness fields is inherently non-unique, as distinct spatial configurations often yield nearly identical sensor responses under limited measurements [24, 25].

These observations indicate that the difficulty of continuous stiffness inversion arises not only from the expressive capacity of the learning model, but also from the choice of field representation and the formulation of the inverse mapping. Therefore, we reformulate this continuous stiffness inversion as a latent representation estimation problem. By parameterizing the continuous stiffness field through a Karhunen–Loève (KL) expansion derived from a spatial covariance kernel, the inverse mapping is performed within a compact latent subspace induced by the covariance structure. This formulation constrains poorly identifiable spatial variations through the covariance structure while bypassing intermediate modal extraction to preserve broadband measurement information. Our key contributions are summarized as follows:

- **Representation-Based Inverse Reformulation:** The continuous stiffness inversion problem is reformulated from unconstrained field regression into latent coefficient estimation within a covariance-induced subspace.
- **Kernel-Induced Statistical Representation:** A KL expansion derived from the spatial covariance kernel is employed to represent continuous stiffness fields, defining both the latent variables and the admissible solution space.
- **Direct Observation-to-Representation Mapping:** Sparse FRFs are directly mapped to latent field coefficients without requiring modal feature extraction.

2 Methodology

In this work, we parameterize the continuous stiffness field using a kernel-induced Karhunen–Loève (KL) expansion. This spatially correlated representation is particularly suited for continuous, early-stage diffuse degradation (e.g., material fatigue or environmental aging), where elastic properties vary smoothly over space rather than exhibiting discrete macro-fractures. Within this unified formulation, the KL-based mathematical representation simultaneously defines: (i) the mathematical representation of the continuous field, (ii) the latent coefficient vector used as the learning target, and (iii) a covariance-induced subspace constraint defining the admissible solution space. Consequently, continuous parameter inversion is transformed into estimating a compact latent coefficient vector α , from which the continuous stiffness field $E(x)$ is deterministically reconstructed.

2.1 Forward dynamic mapping from stiffness fields to FRFs

The relationship between the continuous stiffness field and the measured dynamic response is established through the structural governing equation. For an Euler–Bernoulli beam with a spatially varying effective Young’s modulus $E(x)$, the transverse vibration satisfies:

$$\frac{\partial^2}{\partial x^2} \left(E(x) I \frac{\partial^2 w(x, t)}{\partial x^2} \right) + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = f(x, t), \quad (1)$$

where $w(x, t)$ denotes the transverse displacement, I is the second moment of area, ρA represents the mass per unit length, and $f(x, t)$ denotes the external excitation. This equation shows that the spatial stiffness distribution directly affects the structural restoring force through the bending rigidity $E(x)I$.

After spatial discretization and transformation into the frequency domain, the dynamic equilibrium can be expressed as:

$$(K(E) - \omega^2 M + i\omega C) X(\omega; E) = F(\omega), \quad (2)$$

where $K(E)$ is the stiffness matrix determined by the continuous field $E(x)$, M and C denote the mass and damping matrices, respectively, and $X(\omega; E)$ represents the complex displacement response. The frequency-dependent displacement response is therefore governed by the stiffness-dependent dynamic operator:

$$X(\omega; E) = (K(E) - \omega^2 M + i\omega C)^{-1} F(\omega). \quad (3)$$

The measured displacement frequency response function between an excitation location a and a sensor location s is defined as:

$$H_{s,a}(\omega; E) = \frac{X_s(\omega; E)}{F_a(\omega)}. \quad (4)$$

Therefore, variations in the spatial stiffness field modify the dynamic operator and subsequently alter the frequency-dependent response characteristics, including resonance behavior and response amplitudes. The measured FRFs thus provide indirect observations of stiffness-induced dynamic signatures, establishing the physical basis for formulating continuous stiffness reconstruction as an inverse problem.

2.2 Statistical representation of continuous stiffness fields via KL expansion

Although direct inversion from sparse FRFs preserves broadband dynamic information, reconstructing a continuous stiffness field remains a highly ill-posed inverse problem because the unknown field contains infinitely many spatial degrees of freedom. Meanwhile, intermediate feature extraction, such as modal parameter identification, introduces an additional representation bottleneck that may discard localized stiffness-related response variations. Therefore, instead of directly regressing the continuous field, we reformulate the inverse problem as the estimation of a low-dimensional latent representation.

To construct this representation, a covariance-induced Karhunen–Loève (KL) expansion is introduced to parameterize the continuous stiffness field. The resulting latent space not only defines the learning variables but also constrains the admissible spatial variations according to the prescribed covariance structure. Specifically, the latent field $d(x)$ is parameterized via a truncated KL expansion [26]:

$$d(x) = \sum_{i=1}^N \alpha_i \phi_i(x) \quad (5)$$

where $\phi_i(x)$ denotes the deterministic orthogonal basis functions obtained from the eigendecomposition of the spatial covariance operator, and α represents the truncated latent coefficient vector serving as the explicit learning target. Unlike the conventional KL formulation where the scaling of the eigenvalue is explicitly represented by $\sqrt{\lambda_i}$, the scaling dependent on covariance is incorporated into α_i in this work. Therefore, α directly represents the coordinates of the stiffness field within the covariance-induced basis and serves as the latent variable for inverse learning.

To enforce the prescribed bounds on the material property, the continuous stiffness field $E(x)$ is evaluated from the latent field $d(x)$ through a deterministic mapping operator $\mathcal{M}(\cdot)$:

$$E(x) = \mathcal{M}(d(x)) \quad (6)$$

where $\mathcal{M}(\cdot)$ applies the nonlinear mapping that bounds the resulting stiffness values within a prescribed range.

The truncation of the KL expansion transforms the continuous function space into a finite-dimensional covariance-induced representation space. High-frequency variations that are weakly supported by the covariance structure are excluded, while dominant spatial correlations are preserved [27] ($N = 6$ retains over 90% of the covariance variance, see the Appendix A). The correlation length l_c further controls the spatial regularity of the reconstructed stiffness field by defining the characteristic scale of admissible variations [28], with an extended sensitivity analysis and out-of-distribution evaluations across varying l_c provided in the Appendix B.

2.3 Covariance-induced latent inversion framework

The proposed inversion architecture is illustrated in Figure 1. Based on the forward dynamic relationship and the KL-based field representation established above, sparse dynamic FRFs are mapped into the KL coefficient space through a neural inference model $g_\theta : \mathbf{X} \mapsto \hat{\alpha}$. The predicted continuous stiffness field is subsequently reconstructed through the analytical KL representation, where the KL decoder is fully determined by the prescribed covariance-induced basis.

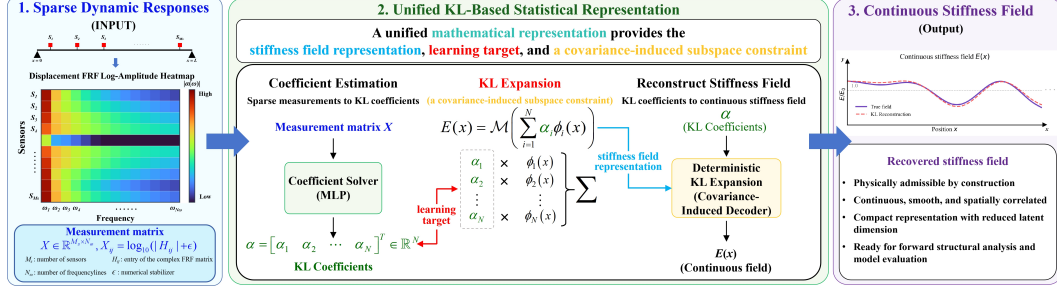


Figure 1: Overview of the inversion framework. Sparse dynamic FRFs are mapped into a low-dimensional KL latent space via a learning module, from which a covariance-driven decoder reconstructs the continuous stiffness field.

3 Experiments

Experiments are conducted on a simply supported Euler–Bernoulli beam, and the KL expansion is truncated to six modes using a correlation length of $l_c = 0.2L$. Stiffness fields are generated independently of the KL representation, details are provided in Appendix D. Independent training and validation follow the procedure described in Section 2.

3.1 Representation comparison

To evaluate the effect of field representation, the KL formulation was benchmarked against direct field regression and a 6-mode DCT baseline under identical learning configurations (Appendix E). As illustrated in Figure 2, the KL-based parameterization consistently reconstructs spatial stiffness variations across different samples, while direct regression and DCT fail to capture the underlying field distributions. These results demonstrate that the KL representation effectively constrains the continuous stiffness field, enabling stable inversions from sparse dynamic observations.

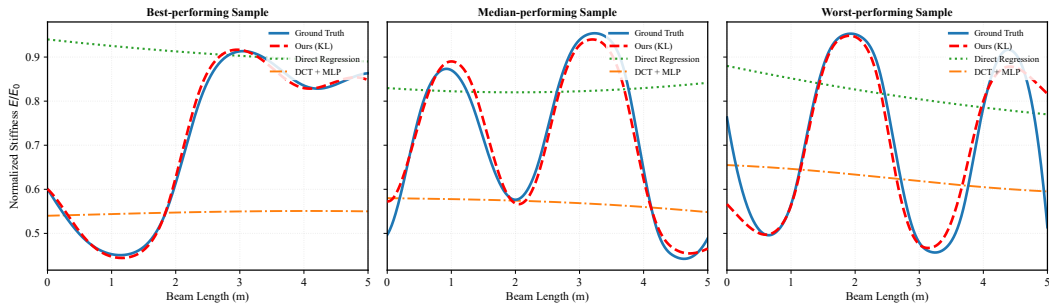


Figure 2: Reconstruction performance across field representations. The proposed KL formulation consistently recovers continuous stiffness profiles (representative $R^2 = 0.9965, 0.9724, 0.7564$), outperforming degraded direct regression and DCT baselines.

3.2 Robustness against observation degradation

The robustness of this proposed representation is evaluated under varying observation conditions, including measurement noise levels from 0% to 20% and sensor configurations with $M_s = 5, 10, 15$. Reconstruction results and quantitative metrics are summarized in Figure 3.

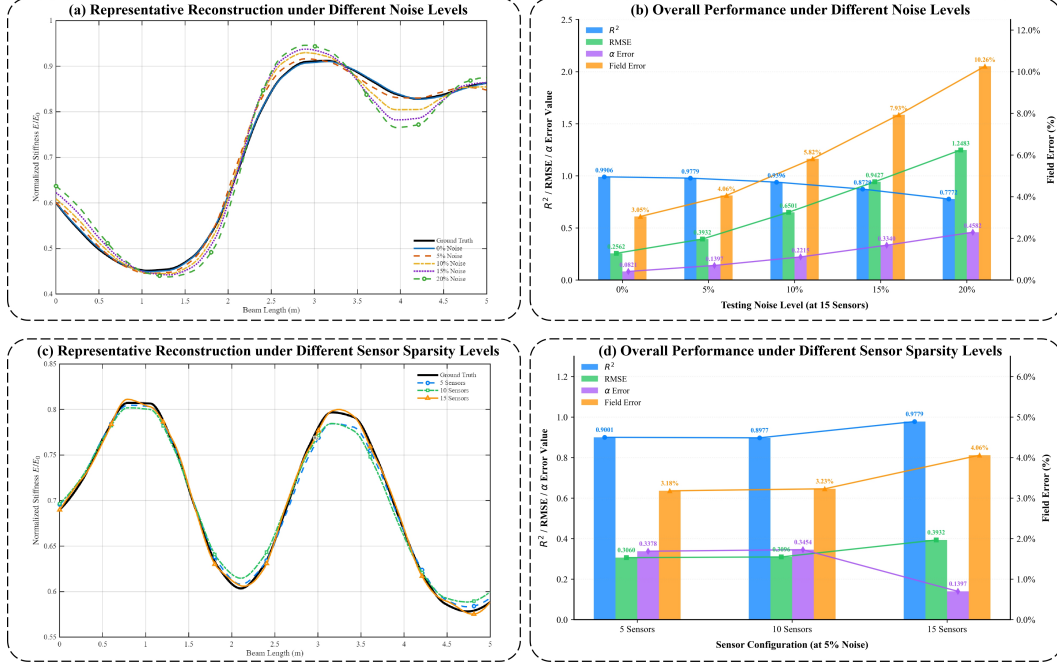


Figure 3: Robustness under degraded observation conditions. Stiffness field reconstruction remains consistent under increasing measurement noise and reduced sensor density, indicating the stability of the proposed KL-based inversion framework.

As shown in Figure 3, the proposed framework maintains consistent stiffness field reconstruction under increasing measurement noise and reduced sensor availability. The reconstruction error increases gradually with the intensity of the noise, whereas denser sensor configurations improve the recovery of localized spatial variations.

4 Discussion

This study highlights the role of representation design in continuous field inversion, showing that covariance-induced latent parameterization can improve inversion stability without increasing neural network capacity. Framing stiffness reconstruction as structured latent representation learning helps mitigate two critical challenges in sparse-observation field inversion: observation information loss from modal feature extraction and spatial instability from unconstrained field regression. The KL-based latent space restricts the inverse solution to a covariance-induced subspace that regularizes the ill-posed mapping, while direct mapping from broadband FRFs retains broadband dynamic information from the original measurements. Pre-structuring the representation space thus provides a principled strategy for learning continuous physical fields under extreme observation sparsity.

The physical scope of this framework is restricted to degradation that manifests itself as a continuous and smoothly varying stiffness reduction, as may occur in certain early-stage fatigue or aging scenarios. In this regime, a covariance-based statistical inductive bias is introduced to constrain noise-induced spatial oscillations. However, this representation limits expressiveness for localized discontinuities, such as discrete macro-fractures or sharp stiffness drops. Future work will extend this fixed covariance specification toward multi-scale or learnable field representations, enabling the latent subspace to better accommodate localized damage features while improving robustness under sparse dynamic observations.

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A Justification for the KL truncation dimension ($N = 6$)

This section provides a quantitative justification for selecting $N = 6$ as the latent truncation dimension in the proposed inversion framework. Setting $N = 6$ establishes a compact yet expressive representation that balances continuous field fidelity with the efficiency of the target search space.

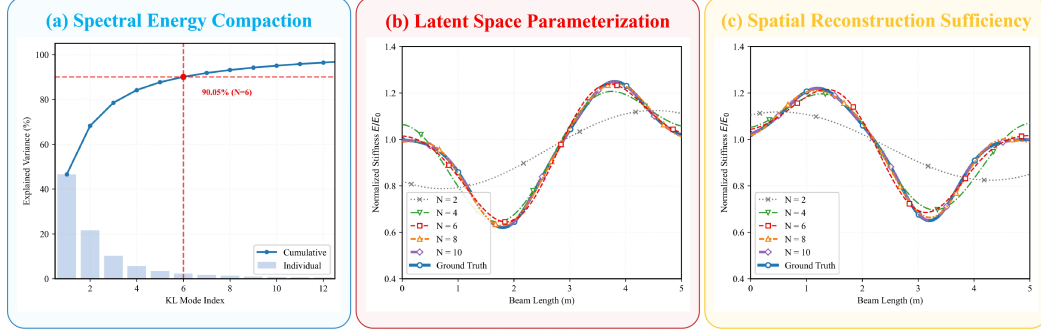


Figure 4: Quantitative justification for the KL truncation dimension ($N = 6$). (a) Spectral Energy Compaction: Cumulative variance spectrum showing that $N = 6$ retains 90.05% of the variance defined by the prescribed covariance kernel under $l_c = 0.2L$. (b) Latent Space Parameterization: Forward latent encoding fidelity across truncation orders for a random target profile, demonstrating accurate field projection at $N = 6$. (c) Spatial Reconstruction Sufficiency: Spatial decoding convergence across truncation orders for another random target profile, illustrating $N = 6$ provides sufficient spatial expressiveness for the tested smooth profiles.

Spectral energy compaction ($> 90\%$): As shown in Figure 4(a), the cumulative variance spectrum indicates that six orthogonal KL modes ($N = 6$) capture 90.05% of the variance of the prescribed covariance-induced representation to the spatial correlation scale ($l_c = 0.2L$). Truncating at $N = 6$ successfully retains the dominant variance of the prescribed mathematical representation, establishing a sufficiently expressive basis for low-dimensional parameterization.

Forward latent encoding fidelity & complexity trade-off: Figure 4(b) evaluates the forward encoding capability across truncation orders $N \in \{2, 4, 6, 8, 10\}$ for a random target profile. Lower orders ($N = 2$ and $N = 4$) exhibit obvious spatial underfitting, failing to resolve localized but smoothly varying stiffness gradients. In contrast, the setting $N = 6$ smoothly captures continuous stiffness variations and aligns closely with the ground truth. Crucially, further increasing the order to $N = 8$ or 10 yields negligible improvements in representation fidelity, while unnecessarily inflating the parameter dimension for downstream inversion. Thus, $N = 6$ delivers the optimal compromise between encoding accuracy and inversion complexity.

Spatial reconstruction sufficiency for smooth fields: The practical sufficiency of $N = 6$ is further confirmed by spatial decoding for another random target profile in Figure 4(c). Expanding the basis to $N = 6$ accurately recovers multi-peak but smoothly varying localized stiffness variations throughout the span. The reconstructed spatial profiles for $N = 8$ and $N = 10$ virtually overlap with that of $N = 6$, confirming that $N = 6$ provides sufficient spatial expressiveness for continuous diffuse degradation in the early stages. Higher-order modes add redundant complexity without enhancing the representation of smoothly varying stiffness fields.

B Covariance kernel specification and sensitivity to spatial correlation mismatch

B.1 Mathematical definition and spectral decomposition of the covariance kernel

To represent the continuous spatial field of elastic modulus reduction in structural elements, an Exponential covariance kernel, associated with the stationary Ornstein-Uhlenbeck process, is formulated over the discretized spatial domain $x \in [0, L]$ [29, 28]:

$$C(x_i, x_j) = \exp\left(-\frac{|x_i - x_j|}{l_c}\right) \quad (7)$$

where x_i and x_j denote the spatial coordinates along the beam model, $|x_i - x_j|$ represents the Euclidean spatial distance, and l_c is the prescribed spatial correlation length scale. In the baseline formulation, a correlation scale of $l_c = 0.2L$ is enforced.

Unlike Gaussian kernels that enforce C^∞ infinite differentiability, the Exponential kernel in Eq. (7) enforces mean-square spatial continuity while accommodating non-smooth local gradient variations[29, 28]. We therefore adopt the Exponential kernel as a prescribed statistical covariance model that permits continuous fields with relatively short spatial correlation scales.

Evaluating Eq. (7) in N_e discrete element locations yields a symmetric positive-definite covariance matrix $\mathbf{C} \in \mathbb{R}^{N_e \times N_e}$. The orthogonal KL basis vectors $\phi_k \in \mathbb{R}^{N_e}$ and the corresponding eigenvalues λ_k are obtained by matrix spectral decomposition [30, 31]:

$$\mathbf{C}\phi_k = \lambda_k\phi_k, \quad k = 1, \dots, N_e \quad (8)$$

Ordering the eigenvalues in descending magnitude ($\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N_e}$), the basis matrix is truncated to the first dominant modes N , forming $\Phi = [\phi_1, \phi_2, \dots, \phi_N] \in \mathbb{R}^{N_e \times N}$ [27]. Each basis vector is normalized to unit length ($\|\phi_k\|_2 = 1$).

To construct learning targets during data set generation, an independent stochastic field $\tilde{\mathbf{d}}$ —generated by harmonic superposition [32]—is projected onto the subspace spanned by Φ via a linear least-squares projection:

$$\alpha = \Phi^\dagger \tilde{\mathbf{d}} = (\Phi^T \Phi)^{-1} \Phi^T \tilde{\mathbf{d}} \quad (9)$$

The bounded elastic modulus field $E(x)$ is subsequently obtained by a sigmoid mapping $D(x) = [1 + \exp(-\tilde{d}(x))]^{-1}$ and property scaling $E(x) = E_0[1 - 0.6D(x)]$, bounded within $[0.2E_0, E_0]$.

B.2 Sensitivity analysis under matched spatial correlation ($l_{c,\text{train}} = l_{c,\text{test}}$)

To evaluate the fundamental capacity of the continuous stiffness inversion framework across varying spatial heterogeneity scales, we first examine the benchmark scenario where the prescribed correlation scale of the KL covariance kernel matches the correlation scale prescribed for the KL representation and the corresponding field-generation setting ($l_{c,\text{train}} = l_{c,\text{test}} \in [0.1L, 0.8L]$).

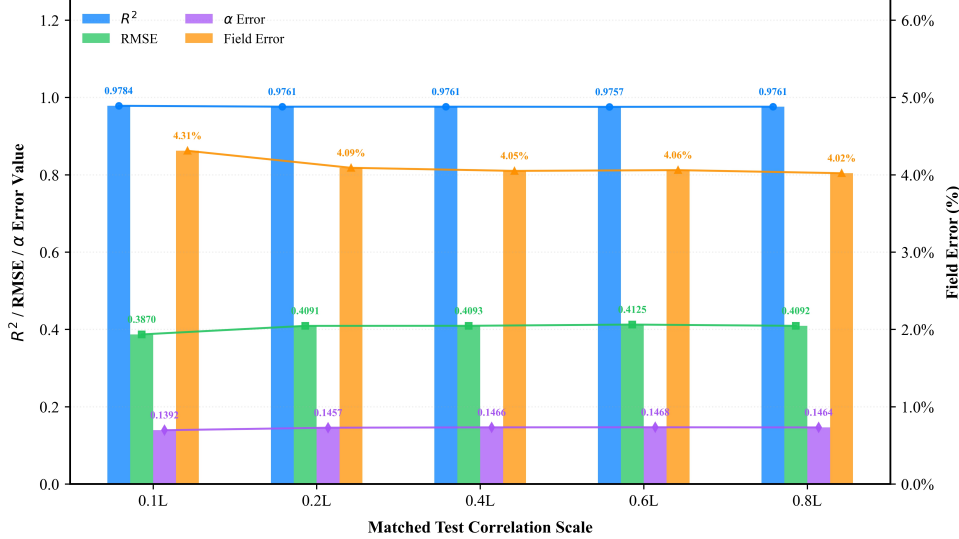


Figure 5: Inversion performance under matched spatial correlation scales ($l_{c,\text{train}} = l_{c,\text{test}}$) across varying correlation scales ($0.1L$ to $0.8L$) at 5% observational noise. Bar chart heights denote R^2 score, RMSE, α Error, and Field Error. Quantitative metrics demonstrate robustness across the tested correlation-length range ($R^2 \in [0.9757, 0.9784]$, Field Error $\in [4.02\%, 4.31\%]$).

As illustrated in Fig. 5, the inversion framework demonstrates robustness across the tested correlation-length range. When the prescribed covariance kernel has a correlation scale consistent with that of the target random field, the resulting KL basis is better aligned with the dominant spatial correlation

structure of the target field. Consequently, both identification of the latent KL coefficients (α Error) and the precision of the field reconstruction (RMSE, Field Error) remain stable across the tested spatial correlation scales, confirming that the covariance-induced reconstruction mapping maintains consistent fidelity across fine-grained ($0.1L$) and highly smooth ($0.8L$) heterogeneity distributions.

B.3 Reconstruction performance under mismatched spatial correlation ($l_{c,\text{train}} \neq l_{c,\text{test}}$)

In practical non-destructive evaluation and structural health monitoring, the characteristic spatial correlation scale and damage topology of target structural degradation are generally unknown [33, 34]. To establish the operational limits of the inversion framework under covariance-kernel misspecification, the baseline model trained with a fixed spatial covariance scale ($l_{c,\text{train}} = 0.2L$) is evaluated against test datasets spanning unseen correlation lengths ($l_{c,\text{test}} \in [0.1L, 0.8L]$) under 5% Gaussian noise.

When evaluated under spatial correlation-scale mismatch, the independently generated test variation fields are projected onto the fixed KL subspace defined by the training correlation length, $\text{span}(\Phi(l_{c,\text{train}}))$, to construct the corresponding test representations. As shown in Fig. 6, the performance metrics reveal two distinct behavioral regimes:

- **Generalization Subspace** ($0.1L \leq l_{c,\text{test}} \leq 0.6L$): High reconstruction fidelity is maintained across this domain ($R^2 \geq 0.9606$, Field Error $\leq 5.03\%$). Moderate correlation mismatches neither destabilize the modal projection nor compromise the neural inversion mapping.
- **Out-of-Distribution (OOD) Failure Boundary** ($l_{c,\text{test}} = 0.8L$): When subjected to ultra-smooth target fields ($l_{c,\text{test}} = 0.8L$), the inversion framework reaches a sharp failure boundary. The coefficient of determination drops to $R^2 = -0.80$, accompanied by a steep increase in continuous field error to 29.8%.

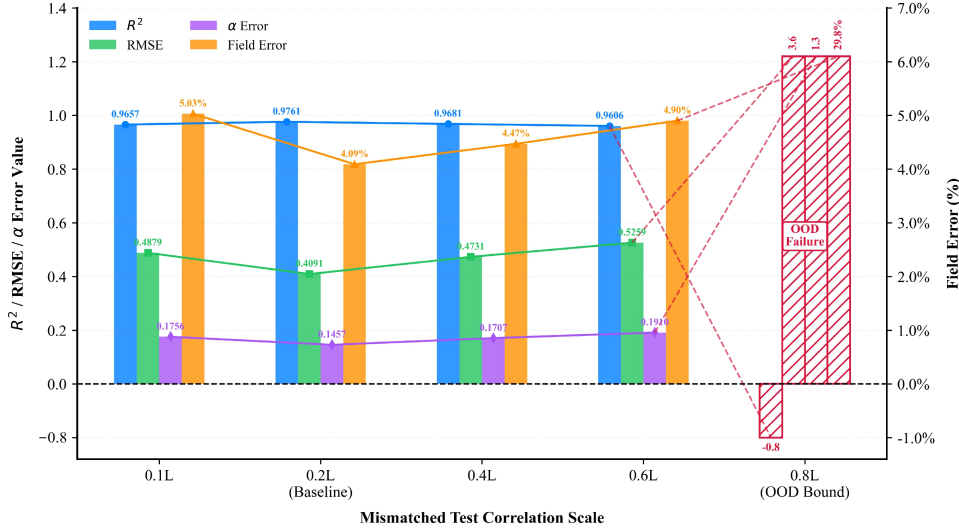


Figure 6: Generalization capability and OOD operational boundary for a fixed spatial correlation scale ($l_{c,\text{train}} = 0.2L$) under mismatched test correlation scales ($l_{c,\text{test}} \in [0.1L, 0.8L]$) at 5% noise. Stable performance is preserved within $0.1L \leq l_{c,\text{test}} \leq 0.6L$ ($R^2 \geq 0.9606$, Field Error $\leq 5.03\%$). At $l_{c,\text{test}} = 0.8L$, the system hits the OOD failure boundary (red-hatched truncated bars); R^2 drops to -0.80 , while field metrics exceed primary visual bounds (RMSE = 3.6, α Error = 1.3, Field Error = 29.8%).

This asymmetric degradation between more localized ($l_{c,\text{test}} < l_{c,\text{train}}$) and ultra-smooth ($l_{c,\text{test}} > l_{c,\text{train}}$) target fields is governed by two computational and covariance-informed mechanisms:

- **Covariance mismatch and subspace misalignment:** In this setting, the truncated KL representation behaves as a covariance-informed low-pass constraint[27]. For localized

target fields ($l_{c,\text{test}} = 0.1L$), the baseline KL representation ($0.2L$) effectively filters out high-frequency spatial noise while preserving the global damage profile ($R^2 \approx 0.9657$). Conversely, when ultra-smooth fields are fitted ($l_{c,\text{test}} = 0.8L$), the target energy is concentrated almost entirely in the dominant first mode. Forcing this smooth response onto a subspace tuned for shorter wavelengths imposes an artificial low-rank over-constraint, distorting the spatial energy allocation across the truncated modes.

- **Latent Space Distribution Shift (OOD Breakdown):** From a data-driven perspective, the neural network is trained to map dynamic responses to latent KL coefficients α defined by the $l_{c,\text{train}} = 0.2L$ representation. Projecting ultra-smooth target fields with $l_{c,\text{test}} = 0.8L$ onto the baseline $l_{c,\text{train}} = 0.2L$ representation suggests a shift in the latent representation encountered by the network relative to that encountered during training. This latent representation shift is reflected by the large coefficient prediction error α Error = 1.3 and ultimately leads to severe field reconstruction degradation. This suggests a substantial shift in the latent representation encountered by the network, which may contribute to the observed degradation in coefficient prediction and the resulting catastrophic field reconstruction error ($R^2 < 0$).

Consequently, while specifying a moderate spatial correlation length ($l_c = 0.2L$) acts as an effective implicit regularizer against observational noise and high-frequency spatial fluctuations, it defines a clear operational boundary beyond which ultra-smooth, large-scale degradation cannot be reliably extrapolated without retraining the model or adapting the KL representation.

C Log-FRF transformation

C.1 Role of Log-amplitude scaling

Structural frequency response functions (FRFs) inherently exhibit extreme dynamic range disparities, typically spanning over six orders of magnitude between anti-resonance minima ($\sim 10^{-6}$) and resonance peaks ($\sim 10^2$) due to light modal damping [1, 13]. In data-driven physical field inversion regimes [21, 35], unscaled input features often lead to severe optimization difficulties. Standard linear or power-law transformations (e.g., square-root scaling) fail to compress such multi-order exponential dynamic ranges, leaving anti-resonance features compressed into near-zero numerical values where their localized structural sensitivities are effectively lost.

To regularize the optimization landscape while preserving spectral fidelity, we enforce an element-wise logarithmic amplitude transformation defined as:

$$X_{ij} = \log_{10}(|H_{ij}| + \epsilon) \quad (10)$$

where H_{ij} represents the complex frequency response between degrees of freedom i and j , and $\epsilon > 0$ denotes a small positive constant ($\epsilon = 10^{-8}$) ensuring numerical stability near anti-resonance nulls.

Formally, this mapping aligns with the decibel (dB) and Bode-plot representations widely adopted in classic structural dynamics and vibration-based feature extraction [13, 36], transforming the exponential scale ratio into a well-conditioned compressed dynamic-range feature domain. From a differential optimization perspective, the scaling derivative $\frac{d}{d|H_{ij}|} \log_{10}(|H_{ij}| + \epsilon) \propto (|H_{ij}| + \epsilon)^{-1}$ naturally dampens extreme gradient vectors at resonance limits while amplifying feature sensitivities within low-amplitude anti-resonance regimes. Consequently, this transformation homogenizes gradient contributions across all spectral frequencies, enabling robust neural network convergence under zero-tuning optimization setups.

C.2 Justification for phase-omitted Log-amplitude representation

Although the complex-valued frequency response function (FRF) encapsulates the phase-lag characteristics governing structural wave propagation, experimental phase measurements exhibit extreme sensitivity to environmental variability and operational measurement noise in practical monitoring setups [22, 23]. For continuous physical field inversion constrained by low-dimensional KL expansions [30], the spatial energy distribution embedded within the amplitude spectrum provides sufficient information to accurately estimate the truncated expansion coefficients under the sensing configuration considered. To evaluate the operational necessity of phase information against

numerical conditioning, an ablation experiment is conducted under an uncalibrated zero-tuning regime ($LR = 10^{-3}$, Adam optimizer, standard MLP architecture). We evaluate two distinct feature representations:

- **Full Complex Feature (X_{complex}):** $X_{\text{complex}} = [\text{Re}(\mathbf{H}), \text{Im}(\mathbf{H})]$, preserving full phase information through decoupled real and imaginary components.
- **Log-Amplitude Feature (X_{log}):** $X_{\text{log}} = \log_{10}(|\mathbf{H}| + \epsilon)$, discarding phase information to prioritize numerical conditioning.

Table 1: Ablation study on different input features under identical training configurations.

Input Feature Type	Phase Information	Training Convergence ($LR = 10^{-3}$)	Mean Field Error
Full Complex (X_{complex})	Preserved	Saturated (Constant Output)	47.12%
Log-Amplitude (X_{log})	Omitted	Stable / Converged	4.06%

Quantitative evaluations summarized in Table 1 indicate that directly processing raw complex components (X_{complex}) results in severe prediction degeneracy. Under test noise levels that span 0% to 20%, the network converges to an approximately invariant output that closely matches the mean of the statistical labels, resulting in a stagnant mean field error of 47.12%. This behavior indicates that the raw complex representation leads to severe optimization difficulties under sparse dynamic observations, likely due to the large and heterogeneous numerical scales across the complex-valued features. Conversely, the logarithmic amplitude representation (X_{log}), which discards phase while compressing the amplitude dynamic range, yields stable convergence and substantially improved reconstruction, achieving a mean field error of 4.06% without apparent numerical instability. These results suggest that, considering the sensing configuration and training protocol, an appropriately scaled amplitude representation provides sufficient information for a robust reconstruction of the continuous KL stiffness field.

D Independence between stiffness field generation and KL representation

To clarify the relationship between the continuous stiffness field generation process and the proposed KL-based representation, the data generation procedure is described separately from the subsequent inverse parameter estimation process. This separation follows the general principle of numerical inverse problems, where the forward model generation and parameter estimation procedure should be clearly distinguished to ensure a consistent evaluation of inversion performance [25]. The general procedure is summarized as shown in Figure 7.



Figure 7: Generation procedure of continuous stiffness fields independent of the KL representation. A continuous spatial variation field is first generated using random Fourier harmonics and subsequently transformed into a damage distribution and Young’s modulus field through nonlinear physical mapping and admissibility constraints. The resulting stiffness field is generated independently, while the underlying spatial variation field is projected onto the KL space afterwards for constructing learning targets.

D.1 Generation of continuous stiffness variation fields

The continuous material variation fields are generated independently of the KL representation used for inverse learning. A spectral representation of a random process is adopted to construct spatially continuous realizations using finite Fourier harmonics [32].

For the one-dimensional beam domain $x \in [0, L]$, the initial variation field is defined as

$$d_{\text{gen}}(x) = \sum_{k=1}^{N_f} \left[a_k \sin\left(\frac{k\pi x}{L}\right) + b_k \cos\left(\frac{k\pi x}{L}\right) \right] \quad (11)$$

where a_k and b_k are independent Gaussian random variables:

$$a_k, b_k \sim \mathcal{N}(0, 1) \quad (12)$$

A limited number of low-frequency harmonics ($N_f = 4$) is adopted to generate smooth spatial variations while avoiding artificial high-frequency oscillations, which is consistent with the continuous stiffness assumption considered in this work. The generated field is subsequently normalized as

$$\tilde{d}(x) = \frac{d_{\text{gen}}(x) - \mu_d}{\sigma_d} \gamma \quad (13)$$

where μ_d and σ_d denote the spatial mean and standard deviation of the generated field, respectively, and $\gamma = 1.5$ controls the magnitude of spatial variation.

D.2 Mapping from spatial variation to continuous stiffness field

The normalized variation field is transformed into a continuous damage distribution through a nonlinear mapping:

$$D(x) = \frac{1}{1 + \exp(-\tilde{d}(x))} \quad (14)$$

The corresponding Young's modulus field is then defined as

$$E(x) = E_0(1 - \eta D(x)) \quad (15)$$

where E_0 denotes the nominal Young's modulus and $\eta = 0.6$ controls the nominal stiffness reduction induced by damage mapping. To account for potential extreme samples and maintain physical admissibility of the material property, the generated stiffness field is further restricted within a predefined range:

$$E(x) = \max(0.2E_0, \min(E_0, E(x))) \quad (16)$$

Through this procedure, the ground-truth stiffness field is obtained from an independent random process followed by a nonlinear physical transformation.

D.3 KL projection for learning target construction

After the independent variation field $\tilde{d}(x)$ has been generated and the corresponding stiffness field $E(x)$ has been constructed, the KL representation is introduced after field generation to construct the low-dimensional learning target. The KL expansion represents a second-order stochastic process in terms of orthogonal eigenfunctions of its covariance operator [26]. Given the KL basis matrix:

$$\Phi = [\phi_1, \phi_2, \dots, \phi_N] \quad (17)$$

the corresponding coefficients are obtained through projection:

$$\alpha = \Phi^\dagger \tilde{\mathbf{d}} \quad (18)$$

where $\Phi^\dagger$ denotes the pseudo-inverse of the KL basis matrix. Therefore, the relationship between the generated field and the KL coefficients is expressed as $\tilde{\mathbf{d}} \rightarrow \alpha$, rather than $\alpha \rightarrow \tilde{\mathbf{d}}$ during data generation.

Consequently, the KL coefficients used in the learning process are extracted from independently generated spatial variation fields underlying the stiffness fields and do not participate in the construction of the ground-truth samples. The KL parameterization can therefore be interpreted as a form of model-space regularization, restricting the inversion problem to a statistically informed subspace rather than solving directly in the full field space [24, 37].

E Optimization behavior under different field representations

To isolate the intrinsic impact of target field representations on the stability of continuous inverse learning, we establish a **Zero-Tuning Stress Test** protocol. By enforcing an unregularized optimization environment devoid of heuristic hyperparameter tuning or adaptive schedules (e.g. learning rate decay, warm-up strategies, or early stopping), optimization artifacts associated with heuristic hyperparameter tuning are minimized, allowing the convergence behavior under a common optimization configuration to be compared across representation spaces. All target architectures are governed by identical, unconstrained Adam optimization parameters:

```
options = trainingOptions("adam", ...
MaxEpochs=200, ... % 200 epochs corresponding to 10,000 iterations
MiniBatchSize=32, ...
InitialLearnRate=1e-3, ... % Fixed learning rate, no adaptive decay
Shuffle="every-epoch", ...
ValidationData={XTest, YTest}, ...
ValidationFrequency=20, ...
Verbose=true);
```

Under this non-regular configuration, the resulting loss trajectories provide a direct probe into the **intrinsic conditioning** and optimization geometry of the representation space itself. The loss dynamics across the candidate field representations is illustrated in Figure 8.

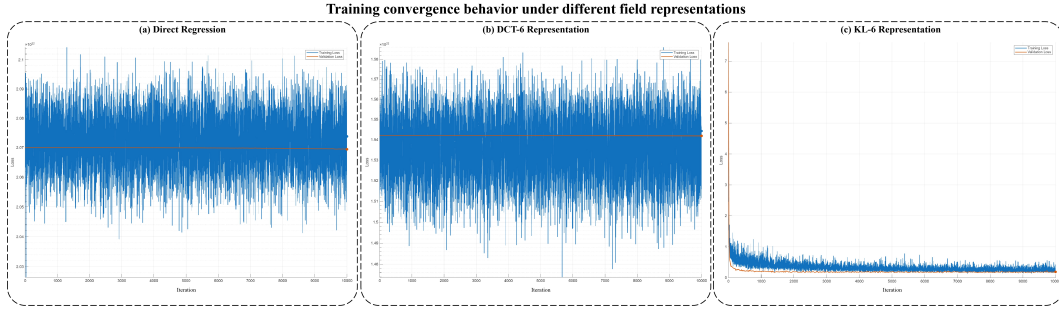


Figure 8: Optimization behavior induced by different stiffness field representations. Training loss curves are presented for (a) direct field regression, (b) DCT-6 coefficient regression, and (c) KL-6 coefficient regression. With the same learning configuration, the KL-based representation achieves stable loss reduction, while the other representations exhibit strong optimization fluctuations.

E.1 Direct regression in the original field space

In the direct field regression baseline, stiffness values across all spatial discretization grid points are treated as mutually independent target variables. Consequently, the neural network is tasked with learning an operator that maps sparse dynamic response measurements directly onto an unconstrained, high-dimensional parameter space lacking spatial coherence constraints. Although output variable normalization mitigates numerical variance across spatial degrees of freedom, the underlying inverse problem remains severely ill-posed and underdetermined due to non-uniqueness wherein distinct spatial stiffness configurations yield virtually identical dynamic signatures. Under the Zero-Tuning Stress Test, this fundamental ill-conditioning manifests itself as high-frequency loss oscillations and persistent divergence during training (Figure 8(a)), demonstrating the intrinsic optimization vulnerability of unconstrained pointwise field representations.

E.2 DCT representation with a general spectral basis

The Discrete Cosine Transform (DCT) introduces a reduced-dimensional spectral target space by projecting the continuous stiffness field onto predefined harmonic cosine functions. However, because these cosine basis functions are field-agnostic, they are constructed independently of the spatial correlation structure specified for the target stiffness field. As a result, the latent coefficient

space does not explicitly incorporate the spatial dependencies encoded by the target field covariance. As shown in Figure 8(b), under the Zero-Tuning Stress Test, the DCT-based representation exhibits persistent optimization fluctuations and fails to achieve stable convergence.

To rule out the possibility that this optimization pathology is caused by an insufficient truncation order ($N = 6$) that fails to capture localized spatial variations, we expanded the dimension of the DCT target to higher orders ($N \in \{10, 12, 15, 20\}$) under the identical zero-tuning protocol. As illustrated in Figure 9, increasing the dimension of the DCT truncation fails to restore the stability of the training. Across all higher-order configurations during 10,000 iterations, the training loss displays irregular large-scale numerical oscillations (ranging up to 10^{21}), while the validation loss remains completely stagnant. This persistent behavior indicates that increasing the dimension of field-agnostic cosine bases alone is insufficient to stabilize inverse learning under sparse dynamic observations, where additional spectral weighting or regularization strategies may be required.

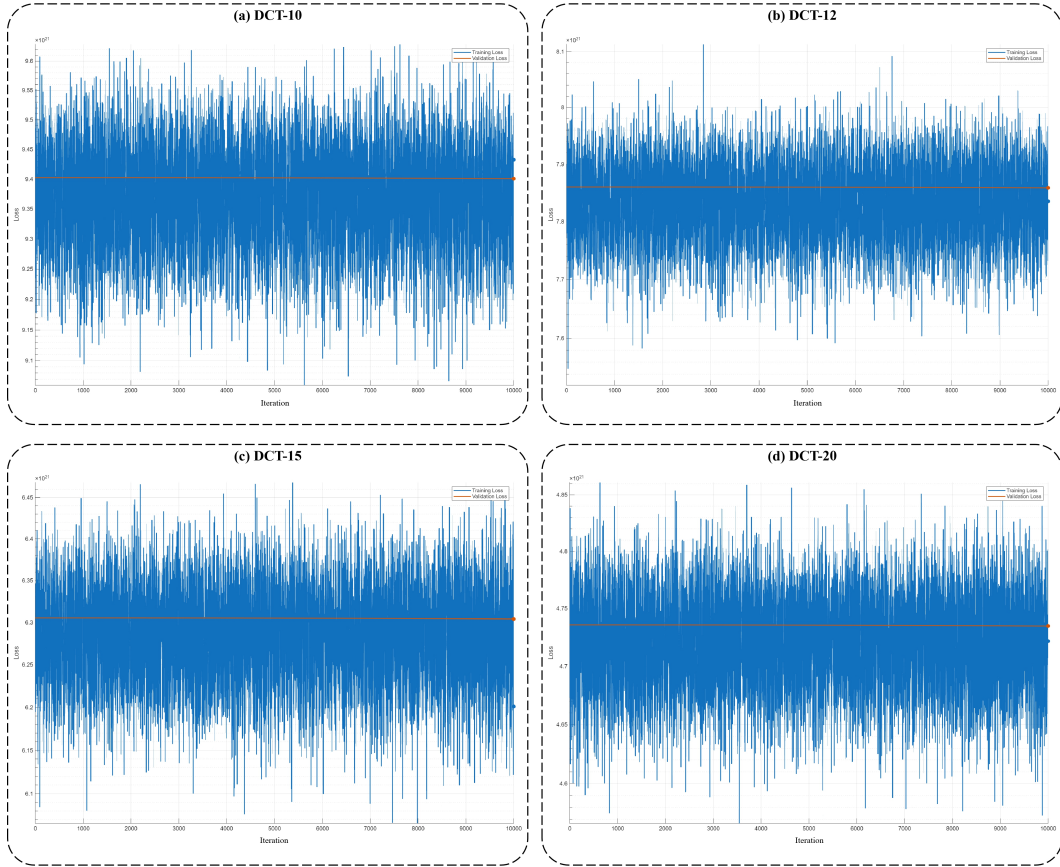


Figure 9: Loss trajectories of high-order DCT representations ($N \in \{10, 12, 15, 20\}$) over 200 epochs (corresponding to 10,000 iterations). Showing unregularized training loss oscillations and flat validation loss.

It is worth clarifying that the failure of DCT under the Zero-Tuning Stress Test does not stem from a deficiency in its mathematical expressiveness; as a complete orthogonal basis, DCT is theoretically capable of approximating arbitrary continuous field distributions. In principle, optimization stability could potentially be improved by introducing explicit spectral weighting strategies—such as Sobolev-type modal loss weighting ($w_k = k^{-p}$) or manually tuned ℓ_2 regularization—to control high-frequency contributions during optimization. However, designing such manual decay schemes introduces additional hyperparameter dependencies and requires explicit assumptions regarding the expected smoothness of the target field. The zero-tuning protocol intentionally excludes these additional interventions to highlight a distinction between representation-induced priors and manually introduced regularization: while generic bases such as DCT require additional spectral weighting or regularization to incorporate smoothness assumptions, the KL expansion directly embeds such

assumptions through the kernel-induced covariance structure and its associated spectral decay. Consequently, KL provides a covariance-informed latent space that is better aligned with the continuous stiffness inversion problem.

E.3 Kernel-induced KL representation

In contrast, the KL expansion parameterizes the stochastic stiffness field using an orthogonal basis derived directly from the spectral decomposition of a spatial covariance kernel. This construction inherently encodes the spatial correlation structure of the continuous parameter field directly into the latent learning space, mapping the continuous parameter inversion to estimate a compact coefficient vector α . By reformulating high-dimensional field reconstruction into latent coefficient regression, the KL representation imposes an implicit covariance-induced subspace constraint on the admissible solution space. This constraint filters out spatial inconsistencies and yields a well-conditioned optimization landscape. As demonstrated by the smooth, monotonic loss trajectory in Figure 8(c), the KL-based representation enables stable, robust convergence without relying on hyperparameter decay, loss regularization, or early stopping. Overall, these comparative dynamics demonstrate that the optimization stability of continuous field inversion is primarily dictated by whether the target representation space aligns with the field’s underlying spatial correlation structure. The observed convergence advantage is therefore associated with the kernel-induced target parameterization rather than changes in neural network capacity.